# MECHANICS.

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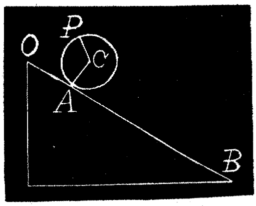
## SOLUTIONS TO PROBLEMS.

7. Proposed by DE VOLSON WOOD, M. A., M. Sc., C. E., Professor of Engineering, Stevens Institute of Technology, Hoboken, New Jersey.

A hollow sphere filled with frictionless water rolls down a rough plane whose length is $l$ and inclination $\theta$; when half way down the water suddenly freezes and adheres to the sphere. Required the time of the descent.

III. Solution by P. H. PHILBRICK, M. S., C. E., Lake Charles, Louisiana.

Let $m$ and $m'$ be the masses of the shell and of the water; $k$ and $k_1$ their radii of gyration about a diameter; and $a$ and $a_1$ the radii of the exterior and interior surfaces of the shell; $F$ the friction on the plane; $t_1$ the time of descent on the upper half of the plane, and $t_2$ the time of descent on the lower half; $V_1$ the velocity immediately before reaching the middle of the plane and $V_2$ the velocity immediately after passing the middle of the plane.




Take the axis of $x$ along the plane positive downward. For motion on the upper half of the plane we have, (see Wood's Analytical Mechanics) for translation, $(m+m')\dfrac{d^2x}{dt^2}=(m+m')g\sin\theta-F\ldots(1)$; and for rotation, $mk^2\dfrac{d^2\theta}{dt^2}=Fa\ldots(2)$, since the water does not rotate. Suppose the point $P$ on the shell to have been at the upper end of the plane upon starting; then $OA=$arc $AP$, or $x=a\theta$; then $dx=ad\theta$ and $d^2x=ad^2\theta$. Multiply (1) by $a^2$, (2) by $a$ and add, substituting $\dfrac{d^2x}{dt^2}$ for $a\dfrac{d^2\theta}{dt^2}$ and we have, $[(m+m')a^2+mk^2]\dfrac{d^2x}{dt^2}=(m+m')a^2g\sin\theta\ldots(4)$.

Integrating gives, $[(m+m')a^2+mk^2]\dfrac{dx}{dt}=(m+m')a^2g\sin\theta t\ldots(5)$.

$\therefore\ \dfrac{dx}{dt}=V=\dfrac{(m+m')a^2g\sin\theta t}{(m+m')a^2+mk^2}\ldots(6)$. Integrating (5) and putting $x=\frac{1}{2}l$ we easily find, $t_1=l^{\frac{1}{2}}\left[\dfrac{(m+m')a^2+mk^2}{(m+m')a^2g\sin\theta}\right]^{\frac{1}{2}}\ldots(7)$. From (6) and (7),

$$V_1=l^{\frac{1}{2}}\left[\frac{(m+m')a^2g\sin\theta}{(m+m')a^2+mk^2}\right]^{\frac{1}{2}}\ldots(8).$$

The energy of the system just before reaching the middle of the plane is, $\frac{1}{2}(m+m')V^2_1+\frac{1}{2}mk^2\left(\frac{V^2_1}{a}\right)$, and just after passing the middle of the plane is, $\frac{1}{2}(m+m')V^2_2+(\frac{1}{2}mk^2+\frac{1}{2}m'k^2_1)\left(\frac{V_2}{a}\right)^2$.

Equating these expressions which must be equal, we have,

$$V_2=V_1\left[\frac{(m+m')a^2+m'k^2}{(m+m')a^2+mk^2+m'k^2_1}\right]^{\frac{1}{2}}\dots(9).$$

Since the ice rotates with the shell, the equations of motion for the lower half of the plane are, $(m+m')\frac{d^2x}{dt^2}=(m+m')g\sin\theta-F\dots(10)$, and

$(mk^2+m'k^2_1)\frac{d^2\theta}{dt^2}=Fa\dots(11)$. Then as before,

$$[(m+m')a^2+mk^2+m'k^2_1]\frac{d^2x}{dt^2}=(m+m')a^2g\sin\theta\dots(12).$$

Integrating gives, $[(m+m')a^2+mk^2+(m+m')k^2_1]\frac{dx}{dt}$

$$=(m+m')a^2g\sin\theta t+C\dots(13).$$

But $\frac{dx}{dt}=V_2$ for $t=0$. $\therefore C=[(m+m')a^2+mk^2+m'k^2_1]V_2$ and (13) becomes, $[(m+m')a^2+mk^2+m'k^2_1]=(m+m')a^2g\sin\theta t+[(m+m')a^2+mk^2+m'k^2_1]V_2\dots(14)$. Integrating, putting $x=\frac{1}{2}l$ and $t=t_2$ gives, $[(m+m')a^2+mk^2+m'k^2_1]l=(m+m')a^2g\sin\theta t^2_2+2[(m+m')a^2+mk^2+m'k^2_1]V_2t_2$.

For brevity, put the coeffecients of $t_2$ and $t^2_2$ equal to $c$ and $d$ respectively, and the absolute term equal to $b$. Then $b=dt^2_2+ct_2$ or $t_2=\frac{1}{2b}(-c+\sqrt{c^2+4bd}\dots(15)$. Equations (7) and (15) give $T=t_1+t_2=$the total time.

**IV. Solution by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy, Ohio University, Athens, Ohio.**

Let $x=$the distance the shell and water of masses $m$, $m'$ respectively have moved down the plane in the time $t$ from the beginning of motion, $F=$ the friction, $r$, $r'$ the exterior and interior radii of the shell, $k$, $k'$ the radii of gyration of $m$ and $m'$, and $\phi=$ the amount of rotation of the shell.

Resolving parallel to the plane, and taking moments about the center of the shell,

$$(m+m')\frac{d^2x}{dt^2}=(m+m')g\sin\theta-F\dots(1),\text{and } mk^2\frac{d^2\theta}{dt^2}=Fr\dots(2).$$

We have also from the geometry, $x=r\phi\dots(3)$. From (3), $\frac{d^2\theta}{dt^2}=\frac{1}{r}\cdot\frac{d^2x}{dt^2}\dots(4)$.

This in (2) gives $mk^2\dfrac{d^2x}{dt^2}=Fr^2\ldots(5)$. Eliminating $F$ from (1) and (5), $[(m+m)r^2+mk^2]\dfrac{d^2x}{dt^2}=(m+m')r^2g\sin\theta\ldots(6)$.

Integrating (6) twice, noticing that initially $\dfrac{dx}{dt}=0, x=0$, we have $[(m+m')r^2+mk^2]\dfrac{dx^2}{dt^2}=2(m+m')r^2g\sin\theta.x\ldots(7)$, and $[(m+m')r^2+mk^2)]x$ $=\frac{1}{2}(m+m')r^2g\sin\theta.t^2\ldots(8)$. When $x=\dfrac{l}{2}$, these give $v=\dfrac{dx}{dt}=$

$$\sqrt{\frac{(m+m')r^2g\sin\theta.l}{(m+m')r^2+mk^2}}\ldots(9)\text{ and }t_1=\sqrt{\frac{[(m+m')r^2+mk^2]l}{(m+m')r^2g\sin\theta}}\ldots(10).$$

The circumstances of motion changing at this point, it is necessary to determine the instantaneous change in $v$ and $\omega$, the latter being the value of $\dfrac{d\phi}{dt}=\dfrac{1}{r}\dfrac{dx}{dt}\ldots(11)$ from (3).

Assuming the principle of the conservation of the moment of momentum as holding here,
$mk^2\omega+(m+m')vr=mk^2\omega'+m'k'^2\omega'+(m+m')v'r\ldots(11)$, $v$ having changed to $v'$, and $\omega$ to $\omega'$. But $v=r\omega$, $v'=r'\omega'$, $k^2=\dfrac{2}{5}\dfrac{r^5-r'^5}{r^3-r'^3}$, $k'^2=\dfrac{2}{5}r'^2\ldots(12)$.

These equations give

$$v'=\frac{2m(r^5-r'^5)+5(m+m')(r^3-r'^3)r^2}{2m(r^5-r'^5)+2m'r'^2(r^3-r'^3)+5(m+m')(r^3-r'^3)}\,v\ldots(13).$$

If $y$=the distance passed over from the middle of the plane after any time $t$, and $\phi'$ the corresponding amount of rotation, we have, resolving as before,

$$(m+m')\frac{d^2y}{dt^2}=(m+m')g\sin\theta-F\ldots(14),\quad(mk^2+m'k'^2)\frac{d^2\phi'}{dt^2}=Fr\ldots(15).$$

We have also $y=r\phi'\ldots(16)$.

Eliminating $F$ from (14) and (15) and using (16), there is $[(m+m')r^2$ $+mk^2+m'k'^2]\dfrac{d^2y}{dt^2}=(m+m')r^2g\sin\theta\ldots(17)$.

Integrating (17) twice, and noticing that initially $\dfrac{dy}{dt}=v'$, $y=0$, we have

$$[(m+m')r^2+mk^2+m'k'^2]y=\tfrac{1}{2}(m+m')r^2g\sin\theta.t^2+[(m+m')r^2+mk^2+m'k'^2]v'\ldots(18).$$

Putting $y=\frac{1}{2}l$, we find the time $t_2$ for the lower half of the plane, and then the required time$=t_1+t_2$.